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# Higher order intertwining approach to quasinormal modes

**T Jana and P Roy**

Physics and Applied Mathematics Unit, Indian Statistical Institute, Kolkata-700 108, India

E-mail: [tapas\\_r@isical.ac.in](mailto:tapas_r@isical.ac.in) and [pinaki@isical.ac.in](mailto:pinaki@isical.ac.in)

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## Abstract

Using higher order intertwining operators we obtain new exactly solvable potentials admitting quasinormal mode (QNMs) solutions of the Klein–Gordon equation. It is also shown that different potentials exhibiting QNMs can be related through nonlinear supersymmetry.

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## 1. Introduction

Quasinormal modes (QNMs) are basically discrete complex frequency solutions of real potentials. They appear in the study of black holes and in recent years they have been widely studied [1]. Interestingly, QNMs have also been found in nonrelativistic systems [2]. However, as in the case of bound states or normal modes (NMs) there are not many exactly solvable potentials admitting QNM solutions and often QNM frequencies have to be determined numerically or other approximating techniques such as the WKB method, phase integral method, etc. Consequently, it is of interest to obtain new exactly solvable potentials admitting such solutions.

In the case of NMs or scattering problems, a number of methods based on intertwining technique, e.g., Darboux algorithm [3], supersymmetric quantum mechanics (SUSYQM) [4], etc, have been used successfully to construct new solvable potentials. Usually, the intertwining operators are constructed using first-order differential operators. However, in recent years intertwining operators have been generalized to higher orders [5–9] and this has opened up new possibilities to construct a whole new class of potentials having nonlinear symmetry. In particular, use of higher order intertwining operator or higher order Darboux algorithm leads to nonlinear supersymmetry.

QNMs are associated with outgoing wave-like behaviour at spatial infinity and unlike normal modes (NMs) the QNM wavefunctions have rather unusual characteristic (for example, wavefunctions diverging at both or one infinity) [10]. Such open systems have been studied

using (first-order) intertwining technique [11]. Recently it has also been shown that the open systems can be described within the framework of first-order SUSY [10]. Here, our objective is to examine whether or not intertwining method based on higher order differential operators can be applied to open systems. For the sake of simplicity, we shall confine ourselves to second-order intertwining operators (second-order Darboux formalism) and it will be shown that the second-order Darboux algorithm can indeed be applied to models admitting QNMs although not exactly in the same way as in the case of NMs. In particular, we shall use the second-order intertwining operator to the Pöschl–Teller potential to construct several new potentials admitting QNM solutions. It will also be shown that such potentials may be related to the Pöschl–Teller potential by second-order SUSY. The paper is organized as follows: in section 2, we present the construction of new potentials using second-order intertwining operators; in section 3, nonlinear SUSY underlying the potentials is shown and finally section 4 is devoted to a conclusion.

## 2. Second-order intertwining approach to quasinormal modes

Two Hamiltonians  $H_0$  and  $H_1$  are said to be intertwined by an operator  $L$  if

$$LH_0 = H_1L. \quad (1)$$

Clearly, if  $\psi$  is an eigenfunction  $H_0$  with eigenvalue  $E$ , then  $L\psi$  is an eigenfunction of  $H_1$  with the same eigenvalue provided  $L\psi$  satisfies required boundary conditions. If  $L$  is constructed using first-order differential operators, then intertwining method is equivalent to Darboux formalism or SUSYQM. In particular, if  $V_0$  is the starting potential and  $L = \frac{d}{dx} + W(x)$ , then the isospectral potential is  $V_1 = V_0 + 2\frac{dW}{dx}$  [4]. Similarly, if  $L$  is generalized to higher orders, then it is equivalent to higher order Darboux algorithm or higher order SUSY.

Let us now consider  $L$  to be a second-order differential operator of the form [7]

$$\begin{aligned} L &= \frac{d^2}{dx^2} + \beta(x)\frac{d}{dx} + \gamma(\beta), \\ \beta(x) &= -\frac{d}{dx} \log W_{i,j}(x), \\ \gamma(\beta) &= -\frac{\beta''}{2\beta} + \left(\frac{\beta'}{2\beta}\right)^2 + \frac{\beta'}{2} + \frac{\beta^2}{4} - \left(\frac{\omega_i^2 - \omega_j^2}{2\beta}\right)^2, \end{aligned} \quad (2)$$

where  $\psi_i$  and  $\psi_j$  are the eigenfunctions of  $H_0$  corresponding to the eigenvalues  $\omega_i^2$  and  $\omega_j^2$  and  $W_{i,j} = (\psi_i\psi_j' - \psi_i'\psi_j)$  is the corresponding Wronskian. Then, the isospectral partner potential  $V_2(x)$  obtained via second-order Darboux formalism is given by

$$V_2(x) = V_0(x) - 2\frac{d^2}{dx^2} \log W_{i,j}(x). \quad (3)$$

The wavefunctions  $\psi_i(x)$  and  $\phi_i(x)$  corresponding to  $V_0(x)$  and  $V_2(x)$  are connected by

$$\phi_k(x) = L\psi_k(x) = \frac{1}{W_{i,j}(x)} \begin{vmatrix} \psi_i & \psi_j & \psi_k \\ \psi_i' & \psi_j' & \psi_k' \\ \psi_i'' & \psi_j'' & \psi_k'' \end{vmatrix}, \quad i, j \neq k. \quad (4)$$

The eigenfunctions obtained from  $\psi_i$  and  $\psi_j$  are given by

$$f(x) \propto \frac{\psi_i(x)}{W_{i,j}(x)}, \quad g(x) \propto \frac{\psi_j(x)}{W_{i,j}(x)}. \quad (5)$$

It may be noted that in the case of normal modes, the new potential would be free of any new singularities if the Wronskian  $W_{i,j}(x)$  is nodeless. This in turn requires that the Wronskian be constructed with the help of consecutive eigenfunctions (i.e.,  $j = i + 1$ ). Also, the eigenfunctions  $f(x), g(x)$  in (5) are not acceptable because they do not satisfy the boundary conditions for the normal modes and in any case they are not SUSY partners of the corresponding states in the original potential. Thus, in the case of normal modes the spectrum of the new potential is exactly the same as the starting potential except for the levels used in the construction of the Wronskian. However, we shall find later that not all of these results always hold in the case of QNMs.

Let us now consider one-dimensional Klein–Gordon equation of the form [10]

$$[\partial_t^2 - \partial_x^2 + V(x)]\psi(x, t) = 0. \quad (6)$$

The corresponding eigenvalue equation reads

$$H\psi_n = \omega_n^2\psi_n(x), \quad H = -\frac{d^2}{dx^2} + V(x). \quad (7)$$

The QNM solutions of equation (7) are characterized by the fact that they are either (1) increasing at both ends II or (2) increasing at one end and decreasing at the other (ID, DI). The wavefunctions decreasing at both ends (DD) correspond to bound states or NMs. In the case of QNMs, the eigenvalues ( $\omega_n^2$ ) may be complex or real and negative. If  $\text{Re}(\omega_n) \neq 0$ , then the SUSY formalism cannot be applied since in that case the superpotential  $W(x)$  becomes complex and consequently one of the partner potential becomes complex. So, we shall confine ourselves to the case when  $\text{Re}(\omega_n) = 0$ , i.e.,  $\omega_n^2$  are real and negative. We would also like to mention that in case equation (7) is to be interpreted as a Schrödinger equation one just has to consider the replacement  $\omega_n^2 \rightarrow E_n$ .

There are a number of potentials which exhibit QNMs. A potential in this category is the inverted Pöschl–Teller potential. This potential is used as a good approximation in the study of Schwarzschild black hole and it is given by

$$V_0(x) = v \operatorname{sech}^2 x. \quad (8)$$

Equation (7) for the potential (8) can be solved in different ways. One of the simplest way is to apply the shape invariance criteria [4] and the solutions are found to be [12–14]

$$\omega_n^\pm = -i(n - A^\pm), \quad A^\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - v} = -\frac{1}{2} \pm q \quad (9)$$

$$\psi_n^\pm(x) = (\operatorname{sech} x)^{(A^\pm - n)} {}_2F_1\left(\frac{1}{2} + q - i\omega_n^\pm, \frac{1}{2} - q - i\omega_n^\pm, 1 - i\omega_n^\pm, \frac{1 + \tanh x}{2}\right). \quad (10)$$

We note that the behaviour of the wavefunctions (10), i.e., whether they represent a NM or QNM depends on the value of the parameter  $v$ . For  $v \in (0, 1/4)$ , i.e.,  $A^+ \in (-1/2, 0)$ ,  $A^- \in (-1, -1/2)$ , the wavefunctions represent outgoing waves and are of the type II. For  $v < 0$  (i.e.,  $A^+ > 0$ ,  $A^- < 0$ ) then the wavefunctions represent NMs when  $n < A^+$  while for  $n > A^+$  they are QNMs. On the other hand, the wavefunctions are always QNMs corresponding to  $\omega_n^-$ . It may be noted that for the QNMs the wavefunctions (10) for even  $n$  are nodeless while those for odd  $n$  have exactly one node at the origin. This behaviour of the wavefunctions is quite different from those occurring in the case of NMs. Using the procedure mentioned above we shall now construct new exactly solvable potentials admitting QNM solutions.

### 2.1. Construction of isospectral partner potential using NMs

*Case 1:  $V_0$  with two NMs.* In order to apply the second-order intertwining approach one may start with a potential  $V_0(x)$  admitting (1) at least two NMs and the rest QNMs or (2) only QNMs. We begin with the first possibility. Thus, we consider  $\nu = -5.04$  so that there are two NMs. In this case, we obtain from (9)  $A^\pm = 1.8, -2.8$ . Thus, the NMs correspond to  $\omega_0^+ = 1.8i, \omega_1^+ = 0.8i$  and are given by

$$\psi_0^+(x) = (\operatorname{sech} x)^{A^+}, \quad \psi_1^+(x) = (\operatorname{sech} x)^{(A^+-1)} {}_2F_1\left(2A^+, -1, A^+, \frac{1 + \tanh x}{2}\right). \quad (11)$$

The QNMs in this case correspond to the frequencies  $\omega_n^+, n = 2, 3, \dots$  and are given by  $\psi_n^+(x)$ . Also for  $\omega_n^-, n = 0, 1, 2, \dots$ , there is another set of QNMs and the corresponding wavefunctions are given by  $\psi_n^-(x)$ . In this case  $A^- < 0$  and consequently there is no NM.

Let us construct a potential isospectral to (8) using the NM frequencies  $\omega_0^+$  and  $\omega_1^+$ . Then, from (10) the Wronskian  $W_{0,1}^+$  is found to be

$$W_{0,1}^+ = -(\operatorname{sech} x)^{2A^+-1}. \quad (12)$$

Clearly,  $W_{0,1}^+$  does not have a zero. In this case, the new potential  $V_2^+(x)$  is free of singularities and is given by

$$V_2^+(x) = V_0(x) - 2 \frac{d^2}{dx^2} \log W_{0,1}^+(x) = -(1 - A^+)(2 - A^+) \operatorname{sech}^2 x. \quad (13)$$

Using the value of  $A^+$  it is easy to see that the new potential  $V_2^+(x)$  in (13) does not support any bound state but only QNMs. This is also reflected by the explicit expressions for the wavefunctions. Using (5) we find

$$f^+(x) = (\operatorname{sech} x)^{(1-A^+)}, \quad g^+(x) = -(\operatorname{sech} x)^{-A^+} \tanh x. \quad (14)$$

From (14) it follows that the above wavefunctions are QNMs corresponding to  $-\omega_1^+$  and  $-\omega_0^+$ , respectively. Note that these two QNMs are new and were not present in the original potential. This, in fact, is where the behaviour of the new potential is different from the usual case. In the case of potentials supporting only NMs, the wavefunctions  $f^+(x), g^+(x)$  obtained through (5) do not have acceptable behaviour. However in the present case both these wavefunctions become QNMs instead of NMs and they have acceptable behaviour at  $\pm\infty$  as can be seen from (14) as well as from figure 1. The other wavefunctions  $\phi_n^+(x), n = 2, 3, \dots$ , corresponding to QNM frequencies  $\omega_n^+ = -i(n - A^+)$  can be obtained using (4) and are given by

$$\begin{aligned} \phi_n^+(x) = & (\operatorname{sech} x)^{(A^+-n)} [(n-1)nF_n \tanh^2 x \\ & + c_1(2n-3)F_{n+1} \operatorname{sech}^2 x \tanh x + c_1c_2F_{n+2} \operatorname{sech}^4 x], \quad n = 2, 3, \dots, \end{aligned} \quad (15)$$

where

$$\begin{aligned} c_1 = & -\frac{n(2A^+ - n + 1)}{2(A^+ - n + 1)}, \quad c_2 = \frac{(-n + 1)(2A^+ - n + 2)}{2(A^+ - n + 2)} \\ F_n = & {}_2F_1\left(-n, 2A^+ - n + 1, A^+ - n + 1, \frac{1 + \tanh x}{2}\right) \\ F_{n+1} = & {}_2F_1\left(-n + 1, 2A^+ - n + 2, A^+ - n + 2, \frac{1 + \tanh x}{2}\right) \\ F_{n+2} = & {}_2F_1\left(-n + 2, 2A^+ - n + 3, A^+ - n + 3, \frac{1 + \tanh x}{2}\right). \end{aligned} \quad (16)$$

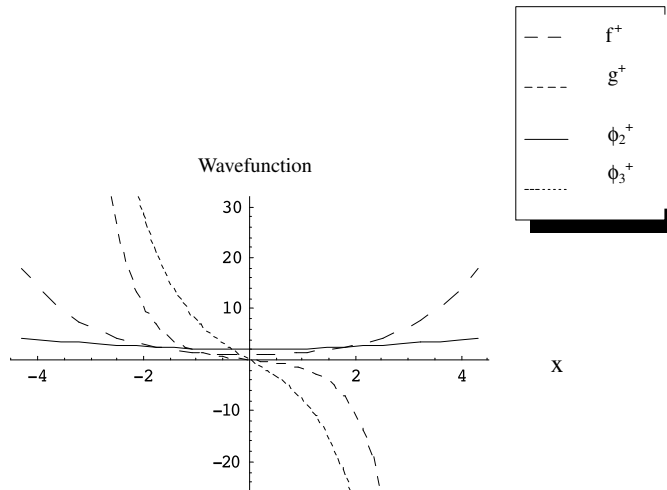


Figure 1. Graph of  $f^+(x)$ ,  $g^+(x)$ ,  $\phi_2^+(x)$  and  $\phi_3^+(x)$  for  $A^+ = 1.8$ .

To see the nature of the wavefunctions (15) we have plotted  $\phi_2^+(x)$  and  $\phi_3^+(x)$  in figure 1. From the figure it can be seen that these wavefunctions are indeed QNMs and for even  $n$  they do not have nodes while for odd  $n$  they have one node at the origin. We would like to point out that the new potential  $V_2^+(x)$  has two more QNMs than  $V_0(x)$ . Thus, except for two additional QNMs, the QNM frequencies  $\omega_n^+$  are common to both  $V_0(x)$  and  $V_2^+(x)$ . We now examine the second set of solutions corresponding to  $\omega_n^-$ . It can be shown by the direct calculation that the new potential (13) also possesses this set of solutions.

Case 2:  $V_0(x)$  with three NMs. Let us now consider the potential (8) supporting three NMs. A convenient choice of the parameter is  $\nu = -6.2$ , so that  $A^+ = 2.04$ ,  $A^- = -3.04$ . We shall now construct the new potential using the NM frequencies  $\omega_1^+ = 1.04i$  and  $\omega_2^+ = 0.04i$ . The Wronskian  $W_{1,2}^+$  is found to be

$$W_{1,2}^+(x) = \frac{(\operatorname{sech} x)^{2A^+-1}}{2(A^+ - 1)} [A^+ - 2 - (A^+ - 1) \cosh 2x]. \tag{17}$$

Now using (3) we obtain

$$V_2^+(x) = -(A^+ - 1)(A^+ - 2) \operatorname{sech}^2 x + 8(A^+ - 1) \frac{(A^+ - 2) \cosh 2x - (A^+ - 1)}{[(A^+ - 1) \cosh 2x - (A^+ - 2)]^2}. \tag{18}$$

To get an idea of the potential, we have plotted  $V_2^+(x)$  in figure 2. From figure 2, it is clear that  $V_2^+(x)$  supports at least one NM. Next to examine the wavefunctions we first consider  $f^+(x)$  and  $g^+(x)$ . From relation (5), we obtain

$$\begin{aligned} f^+(x) &= \frac{2(A^+ - 1)(\operatorname{sech} x)^{-A^+} \tanh x}{(A^+ - 1) \cosh 2x - (A^+ - 2)}, \\ g^+(x) &= \frac{(\operatorname{sech} x)^{-(A^++1)} [1 - (2A^+ - 1) \tanh^2 x]}{(A^+ - 1) \cosh 2x - (A^+ - 2)}. \end{aligned} \tag{19}$$

Also, from (4) it follows that

$$\phi_0^+(x) = \frac{4(A^+ - 1)(\operatorname{sech} x)^{(A^+-2)}}{(A^+ - 1) \cosh 2x + A^+ - 2}. \tag{20}$$

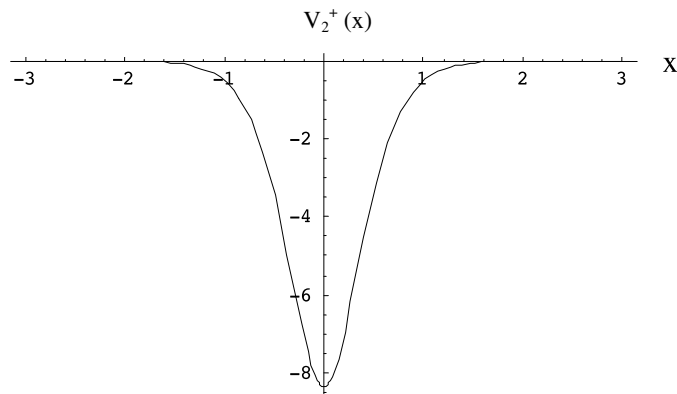


Figure 2. Graph of  $V_2^+(x)$  for  $A^+ = 2.04$ .

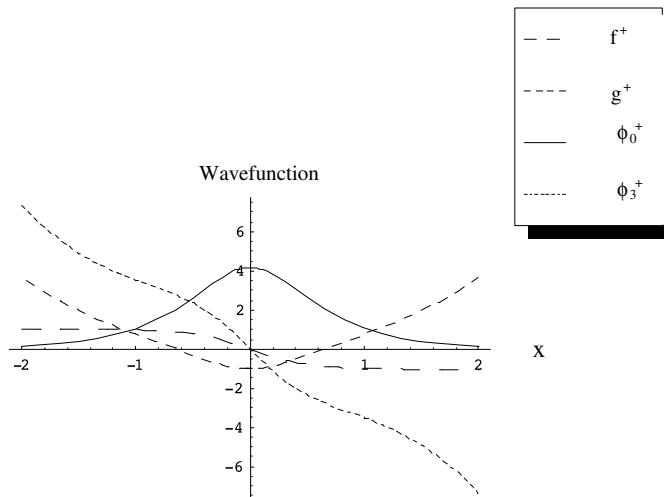


Figure 3. Graph of  $f^+(x)$ ,  $g^+(x)$ ,  $\phi_0^+(x)$  and  $\phi_3^+(x)$  for  $A^+ = 2.04$ .

From (19) it follows that  $f^+(x)$  and  $g^+(x)$  are new QNMs corresponding to frequencies  $-\omega_2^+ = -0.04i$  and  $-\omega_1^+ = -1.04i$ , respectively. The former has one node and the later has two nodes. The nodal structure of the QNM wavefunctions are different from those obtained earlier. The reason for this is that since we started with the first and second excited state NMs and the Wronskian  $W_{1,2}^+$  is nodeless, the behaviour of the original wavefunctions  $\psi_{1,2}^+(x)$  is retained by  $f^+(x)$  and  $g^+(x)$ . However,  $\phi_0^+(x)$  is a NM at  $\omega_0^+ = 2.04i$  and it does not have a node because  $\psi_0^+(x)$  does not have one. Also other QNM wavefunctions  $\phi_n^+(x)$ ,  $n = 3, 4, \dots$  have either no node or one node. In figure 3, we have plotted some of the wavefunctions. We also note that although the potential in (13) is of a similar nature as (8), the potential (18) is of a completely different type. In particular, it is a non-shape-invariant potential. Finally we discuss the possibility of a second set of solutions for the potential (18). We recall that the existence of two sets of solutions for the potential (8) (or (13)) was due to the fact that the parameter  $\nu$  could be expressed as a product of two different parameters  $A^\pm$ . However, in

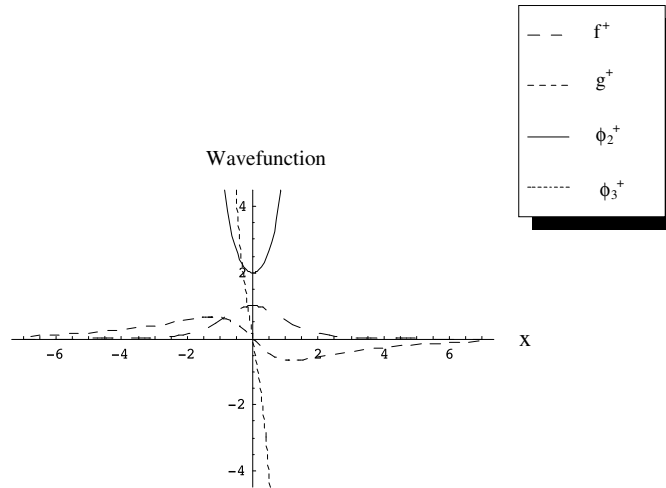


Figure 4. Graph of  $f^+(x)$ ,  $g^+(x)$ ,  $\phi_2^+(x)$  and  $\phi_3^+(x)$  for  $A^+ = -0.4$ .

the case of (18) the entire potential cannot be expressed in terms of two distinct parameters because of the presence of the second term. Consequently, the potential (18) has only one set of solution mentioned above.

2.2. Construction of isospectral partner potential using QNMs

Case 1: Potential based on consecutive QNMs. Here, we shall construct isospectral partner of a potential which has only QNMs. Thus, we consider  $\nu = 0.24$  and in this case  $A^\pm = -0.4, -0.6$ . We consider the  $A^+$  sector and begin with the frequencies  $\omega_0^+$  and  $\omega_1^+$ . In this case, the expression for the Wronskian  $W_{0,1}^+$ , the new potential  $V_2^+(x)$  and the QNM wavefunctions can be derived from expressions (12), (13) and (15). respectively, except that we now have to use a different parameter value. Thus, the new potential is given by

$$V_2^+(x) = -3.36 \operatorname{sech}^2 x. \tag{21}$$

For this potential, the NMs corresponding to  $-\omega_0^+ = 0.4i$  and  $-\omega_1^+ = 1.4i$  are given, respectively, by

$$f^+ = (\operatorname{sech} x)^{1.4}, \quad g^+ = (\operatorname{sech} x)^{0.4} \tanh x. \tag{22}$$

Clearly, these NMs are not SUSY partner of any levels in  $H_0$ . The QNMs correspond to  $\omega_n^+ = -i(n + 0.4), n = 2, 3, \dots$ , and are given by (15) with  $A^+ = -0.4$ . We have plotted some of the wavefunctions in figure 4. From the figure, we find that the wavefunctions  $f^+(x)$  and  $g^+(x)$  correspond to NMs and the other wavefunctions represent QNMs which are the SUSY partners of the QNMs in  $H_0$ . We note that as in (13) the potential (21) has two sets of QNMs, the second of which corresponds to  $\omega_n^-$ .

Case 2: Potential based on non-consecutive QNMs. Here we shall consider the previous parameter values (i.e.,  $A^+ = -0.4$ ) and construct the new potential using the non-consecutive levels  $\omega_0^+$  and  $\omega_3^+$ . In this case, the Wronskian is given by

$$W_{0,3}^+ = \frac{(\operatorname{sech} x)^{(2A^+-3)}}{2(A^+ - 2)} [(9 - 6A^+) \tanh^2 x + 3]. \tag{23}$$



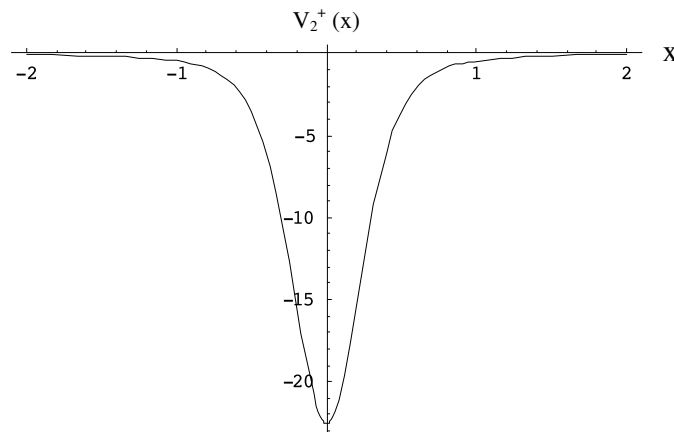


Figure 5. Graph of  $V_2^+(x)$  for  $A^+ = -0.4$ .

It can be shown that the Wronskian (23) is nodeless. Now, using (3) the new potential is found to be

$$V_2^+(x) = \frac{(A^+ - 2)[2(A^+(A^+ - 2)(3A^+ - 7) - 2A^+(A^+ - 1)(A^+ - 4) \cosh 2x - (3 - 2A^+)^2(A^+ - 1) \operatorname{sech}^2 x]}{[1 - A^+ + (A^+ - 2) \cosh 2x]^2}. \quad (24)$$

The potential (24) is free of any singularity and is plotted in figure 5. From figure 5 we find that it supports NMs. As explained earlier, this potential also has one set of solution. We now consider the wavefunctions corresponding to  $\psi_0^+(x)$  and  $\psi_3^+(x)$ . These are obtained from (5) and are given by

$$\begin{aligned} f^+(x) &= \frac{2(A^+ - 2)}{(9 - 6A^+) \tanh^2 x + 3} (\operatorname{sech} x)^{(3 - A^+)}, \\ g^+(x) &= \frac{(1 - 2A^+) \tanh^2 x + 3}{(9 - 6A^+) \tanh^2 x + 3} \sinh x (\operatorname{sech} x)^{(1 - A^+)}. \end{aligned} \quad (25)$$

The above wavefunctions (with zero and one node respectively) represent NMs corresponding to  $-\omega_3^+ = 3.4i$  and  $-\omega_0^+ = 0.4i$ . The other wavefunctions can be obtained through (4). The two QNM wavefunctions lying between  $\omega_0^+$  and  $\omega_3^+$  are  $\phi_{1,2}^+(x)$  corresponding to  $\omega_1^+ = -1.4i$  and  $\omega_2^+ = -2.4i$ . We have plotted these wavefunctions in figure 6. From figure 6, it can be seen that  $f^+(x)$  and  $g^+(x)$  are NMs, while  $\phi_{1,2}^+(x)$  are QNMs with the later having two nodes. The rest of the QNM wavefunctions corresponding to the frequencies  $\omega_n^+ = -(n + 0.4)i$ ,  $n \neq 0, 3$ , are given by  $\phi_n^+(x)$  and they have either zero or one node.

### 3. Polynomial SUSY

In first-order SUSY, the anticommutator  $\{Q, Q^\dagger\}$  of the supercharges is a linear function of the Hamiltonian. On the other hand, in higher order SUSY,  $\{Q, Q^\dagger\}$  is a nonlinear function

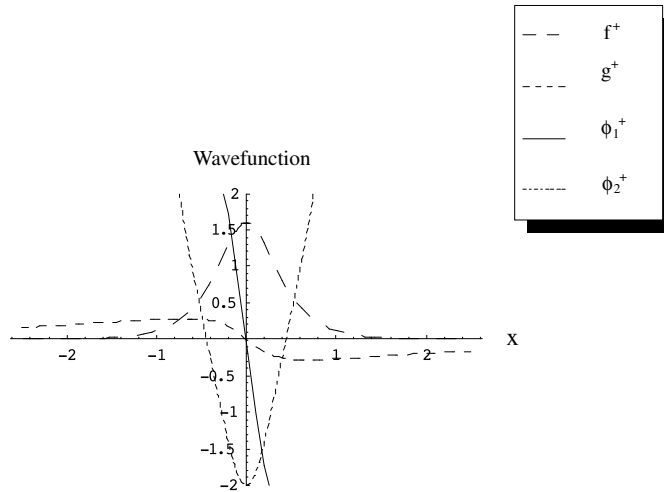


Figure 6. Graph of  $f^+(x)$ ,  $g^+(x)$ ,  $\phi_1^+(x)$  and  $\phi_2^+(x)$  for  $A^+ = -0.4$ .

of the Hamiltonian. It will be shown here that the Hamiltonians  $H_0$  and  $H_2$  are related by second-order SUSY. To this end, we define the supercharges  $Q$  and  $Q^\dagger$  as follows:

$$Q = \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & L^\dagger \\ 0 & 0 \end{pmatrix}, \tag{26}$$

where the operator  $L$  is given by (2).

Clearly, the supercharges  $Q$  and  $Q^\dagger$  are nilpotent. We now define a super Hamiltonian  $H$  of the form

$$H = \begin{pmatrix} H_0 & 0 \\ 0 & H_2 \end{pmatrix}. \tag{27}$$

It can be easily verified that  $Q$ ,  $Q^\dagger$  and  $H$  satisfy the following relations:

$$[Q, H] = [Q^\dagger, H] = 0. \tag{28}$$

Then, the anticommutator of the supercharges  $Q$  and  $Q^\dagger$  is given by a second-order polynomial in  $H$ :

$$H_{ss} = \{Q, Q^\dagger\} = \begin{pmatrix} L^\dagger L & 0 \\ 0 & LL^\dagger \end{pmatrix} = \left(H + \frac{\delta}{2}\right)^2 - c\mathcal{I}, \tag{29}$$

where  $\mathcal{I}$  is the  $2 \times 2$  unit matrix and

$$\delta = -(\omega_i^2 + \omega_j^2), \quad c = \left(\frac{\omega_i^2 - \omega_j^2}{2}\right)^2. \tag{30}$$

Also, we have

$$[Q, H_{ss}] = [Q^\dagger, H_{ss}] = 0. \tag{31}$$

Relations (29) and (31) constitute second-order SUSY algebra.

As an example, let us consider the potentials (8) and (13). The corresponding Hamiltonians  $H_0$  and  $H_2$  are obtained from (7). In this case  $\delta = 0.5416$  and  $c = 0.2916$ , so that from (29) we get

$$H_{ss} = (H + 0.5416)^2 - 0.2916\mathcal{I}. \tag{32}$$

In a similar fashion one may obtain  $H_{ss}$  for the other pair of potentials.

#### 4. Conclusion

Here, we applied the second-order Darboux algorithm to the Pöschl–Teller potential and obtained new exactly solvable potentials admitting QNM solutions. We have considered a number of possibilities to construct the new potentials e.g., starting from NMs or starting from QNMs. It has also been shown that the new potentials are related to the original one by second-order SUSY. We feel, it would also be useful to analyse the construction of potentials using various levels as well as for different values of the parameter  $\nu$  (for example,  $\nu =$  half-integer) [10]. Finally, we believe it would be interesting to extend the present approach to other effective potentials appearing in the study of Reissner–Nordström, Kerr black hole, etc.

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